# Winter mathematics competition—Varna, 1999

**Problem 8.1.** Find all natural numbers $x$ and $y$ such that:

a) $\frac{1}{x} - \frac{1}{y} = \frac{1}{3}$;

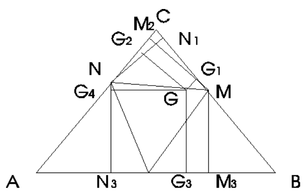
b) $\frac{1}{x} + \frac{1}{y} = \frac{1}{3} + \frac{1}{xy}$.

**Solution:** a) The equation is equivalent to $3y - 3x = xy$, so $x = \frac{3y}{y+3} = \frac{3y+9-9}{y+3} = 3 - \frac{9}{y+3}$. Therefore $y + 3 = 9$ and thus $y = 6$. Hence there is a unique solution $x = 2, y = 6$.

b) Let $x \geq y$ be a solution of the problem. Now $\frac{1}{3} = \frac{1}{x} + \frac{1}{y} - \frac{1}{xy} \leq \frac{2}{y} - \frac{1}{xy} = \frac{2y-1}{xy} < \frac{2y}{xy} = \frac{2}{x}$, giving $x < 6$. When $x = 1$, $x = 2$ or $x = 3$, no solution exists. When $x = 4$, it follows that $y = 9$, and $x = 5$ implies $y = 6$. If $y \geq x$, we apply the same reasoning. The problem has four solutions:

$$x = 4, y = 9; x = 5, y = 6; x = 6, y = 5; x = 9, y = 4.$$

**Problem 8.2.** Given an acute $\triangle ABC$ with centroid $G$ and bisectors $AM(M \in BC), BN(N \in AC), CK(K \in AB)$. Prove that one of the altitudes of $\triangle ABC$ equals the sum of the remaining two if and only if $G$ lies on one of the sides of $\triangle MNK$.




**Solution.** We shall repeatedly use the following property: A segment connecting a vertex of a triangle with a point on the opposite side divides the triangle into two triangles such that the ratio of their areas equals the ratio of the parts into which the point divides the side.

Let $G \in MN$ and $G_1$, $G_2$, $G_3$ be the projections of $G$ on $BC$, $AC$ and $AB$, respectively. Further, denote the projections of $N$ on $BC$ and $AB$ by $N_1$ and $N_3$ and those of $M$ on $AC$ and $AB$ by $M_2$ and $M_3$. We shall prove that $GG_3 = GG_1 + GG_2$ and from the above property it will follow straightforwardly that the altitude from $C$ is equal to the sum of the remaining two altitudes. We obtain

$$\frac{GG_1}{NN_1} = \frac{CM \cdot GG_1}{CM \cdot NN_1} = \frac{S_{GMC}}{S_{NMC}} = \frac{GM}{NM}.$$

By analogy $\frac{GG_2}{MM_2} = \frac{GN}{NM}$, implying that $\frac{GG_1}{NN_1} + \frac{GG_2}{MM_2} = 1$, so

$$GG_1 \cdot MM_2 + GG_2 \cdot NN_1 = MM_2 \cdot NN_1. \quad (1)$$

Let $M_4$ and $G_4$ be the projections of $M$ and $G$ on $NN_3$. It easily follows now that

$$\frac{NN_3 - GG_3}{NN_3 - MM_3} = \frac{GN}{NM} = \frac{GG_2}{MM_2}.$$

Further, using that $NN_3 = NN_1$ and $MM_3 = MM_2$, we obtain $GG_3 \cdot MM_2 = GG_1 \cdot MM_2 + GG_2 \cdot MM_2$ and therefore $GG_3 = GG_1 + GG_2$.

Conversely, let the altitude through $C$ be the sum of the remaining two. Now $GG_3 = GG_1 + GG_2$. If $G^\star = GG_3 \cap MN$, then it follows straightforwardly that the sum of the distances from $G^\star$ to $AC$ and $BC$ equals to $G^\star G_3$. It is easy to check now that $G^\star \cong G$.

**Problem 8.3.** Let $n$ be a natural number. Find all integer values of $m$ such that $k = 2^{m-2}$ is integer and $A = 1999^k + 6$ is a sum of the squares of $n$ integers (not necessarily distinct and different from zero).

**Solution:** It is sufficient to consider only nonnegative values of $m$.

1) $n = 1$ and $k = \dfrac{m}{2}$ is integer only if $m = 4p$ and $m = 4p + 2$. If $m = 4p$, then $A = (2 \cdot 1000 - 1)^{2p} + 1$ and it follows by induction that $A$ is of the form $A = 4a + 7$, so $A$ is congruent to 3 modulo 4. We conclude that $A$ is not a perfect square. If $m = 4p + 2$, then $A = (25 \cdot 80 - 1)^{2p+1} + 6$ and it follows by induction that $A$ is of the form $A = 25a + 5$. Therefore $A$ is not a perfect square, because 5 divides $A$, but 25 does not.

2) $n = 2$ and $k = m$ is integer for any $m$. Now $A = 1999^m + 6$. When $m = 0$, we get $A = 7$, which is not a sum of two squares.

When $m=1$, we obtain $A=2005=41^2+18^2$. If $m\geq 2$, then $A=(2\cdot 999+1)^m+1$ and as above $A$ is congruent to 3 modulo 4. On the other hand the sum of two perfect squares is congruent to 0, 1 or 2 modulo 4 and so no solution exists in this case.

3) $n=2$ and $k=m$ is integer for any $m$. Now $A=(8\cdot 250-1)^{2m}+6$, which can be written in the form $A=8a+7$. Therefore $A$ is congruent to 7 modulo 8, whereas a sum of three perfect squares is congruent to 0, 1, 2, 3, 4, 5 or 6 modulo 8. Thus no solution exists in this case.

4) $n\geq 4$ and $k$ is integer for any $m$. Now $A=(1999^{m2^{n-3}})^2+2^2+1^2+1^2$ and if $a_1=1999^{m2^{n-3}}, a_2=2\cdot a_3=a_4=1, a_5=a_6=\cdots=a_n=0$, then $A=a_1^2+a_2^2+\cdots+a_n^2$.

Answer: if $n=1$, no solution exists;
if $n=2$, there is an unique solution $m=1$;
if $n=3$, no solution exists;
if $n\geq 4$, any $m\geq 0$ is a solution.

**Problem 9.1.** Let $p$ be a real parameter such that the equation $x^2-3px-p=0$ has real and distinct roots $x_1$ and $x_2$.

a) Prove that $3px_1+x_2^2-p>0$.

b) Find the least possible value of

$$A=\frac{p^2}{3px_1+x_2^2+3p}+\frac{3px_2+x_1^2+3p}{p^2}.$$

When does equality obtain?

**Solution:** a) It follows from the equation that $x_2^2 = 3px_2 + p$ and so $3px_1 + x_2^2 - p = 3p(x_1 + x_2) = 9p^2 > 0$. The inequality is strict because otherwise $x_1 = x_2 = 0$.

b) As in a), we obtain $3px_1 + x_2^2 + 3p = 3px_2 + x_1^2 + 3p = 9p^2 + 4p > 0$ (the last inequality follows from the conditions of the problem $x_1$ and $x_2$ to be distinct and real, giving $p \neq 0$). Therefore

$$A = \frac{p^2}{9p^2 + 4p} + \frac{9p^2 + 4p}{p^2} \geq 2$$

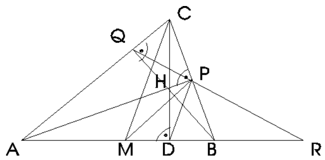
(from the Arithmetic–Geometric Mean Inequality) and equality obtains when $9p^2 + 4p = p^2$, i. e., when $p = -1/2$.

**Problem 9.2.** Given an acute $\triangle ABC$ such that $AC > BC$, let $M$ be the midpoint of $AB$ and let $CD$, $AP$ and $BQ$ be the altitudes. Denote the circumcircle of $\triangle PQC$ by $k_1$ and the circumcircle of $\triangle DRP$ by $k_2$, where $R$ is the point of intersection of $AB$ and $PQ$. Prove that:

a) $MP$ is tangent to both $k_1$ and $k_2$.

b) $RH \perp CM$, where $H$ is the orthocentre of $\triangle ABC$.

**Solution:** a) Note that $H \in k_1$. Since $\angle APM = \angle PAM = 90° - \angle ABC = \angle BCD$, we obtain that $MP$ is a tangent to $k_1$. On the other hand $\angle MPD = \angle MPB - \angle DPB = \angle MPB - \angle DPB = \angle MBP - \angle BPR\,(\triangle ABC \sim \triangle DBP)$, so $\angle ARP = \angle MBP - \angle BPR = \angle MBP - \angle QPC = \angle MBP - \angle BAC(\triangle ABC \sim \triangle PQC)$. Therefore $\angle MPD = \angle MBP$ and thus $MP$ is a tangent to $k_2$.




b) Let $L = CM \cap k_1$. It follows from a) that $ML \cdot MC = MP^2 = MD \cdot MR$. We conclude that $L$ lies on the circumcicrle of $\triangle DRC$ and therefore $RL \perp CM$. Further $HL \perp CM$, since $HC$ is a diameter of $k_1$. Hence $RH \perp CM$.

**Problem 9.3.** A square table filled with nonnegative (not necessarily distinct) integer numbers is said to be a magic square with sum $m$ if the sum of the numbers in each row and each column equals $m$. Prove that the number of magic squares $3 \times 3$ of sum $m$ such that the minimal element among the elements on the main diagonal lies in the centre is $\binom{m+4}{4}$.

**Solution:** It is evident that knowing the elements of main diagonal and the element in the cell $(1, 2)$ (see fig. 1) one can determine all elements in the table. Indeed, there is a unique choice for all remaining cells (see fig. 2). Therefore it suffices to see when all elements are nonnegative and $b$ is the minimal element among the elements on the main diagonal.

| | | |
|---|---|---|
| $a$ | $d$ | |
| | $b$ | |
| | | $c$ |

Fig. 1

| | | |
|---|---|---|
| $a$ | $d$ | $m-a-d$ |
| $m+c-a-b-d$ | $b$ | $a+d-c$ |
| $b+d-c$ | $m-b-d$ | $c$ |

Fig. 2

It is clear from fig. 2 that the following inequalities hold:

(1) $a + d \leq m$;
(2) $b + d \leq m$;

(3) $c \leq a + d$;
(4) $c \leq b + d$;
(5) $a + b + d - c \leq m$.

The conditions of the problem imply $b \leq a$ and $b \leq c$. It is clear now that (3) follows from (4) and (2) and (5) follow from (1). Therefore we can consider only (1) and (4).

Consider the following chain of inequalities

$$b \leq 2b + d - c \leq a + b + d - c \leq a + d \leq m$$

(the first follows from (4), the second from $b \leq a$, the third from $b \leq c$, and the fourth is equivalent to (1)). It is easy to see that knowing the quadruple $(b, 2b+d-c, a+b+d-c, a+d)$ we can uniquely determine $a, b, c$ and $d$ and so find a magic square. Therefore the required number equals the number of quadruples, which is $\binom{m+4}{4}$.

**Problem 10.1.** Find all values of the real positive parameter $a$ such that the inequality $a^{\cos 2x} + a^{2\sin^2 x} \leq 2$ holds for any real $x$.

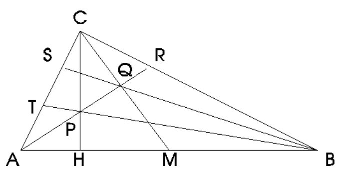
**Solution:** We know that $a^{\cos 2x} + a^{2\sin^2 x} = a^{1-2\sin^2 x} + a^{2\sin^2 x} = \dfrac{a}{a^{2\sin^2 x}} + a^{2\sin^2 x}$. Substitute $t = a^{2\sin^2 x}$. Since $0 \leq \sin^2 x \leq 1$, we obtain that $t$ is between 1 and $a^2$. Our inequality now becomes $\dfrac{a}{t} + t \leq 2 \iff t^2 - 2t + a \leq 0$. Since it holds true for any $x$ (i. e., for any $t$ between 1 and $a^2$), it follows that the roots of $f(t) = t^2 - 2t + a = 0$ lie outside the open interval determined by 1 and $a^2$. Therefore $f(1) \leq 0$ and $f(a^2) \leq 0$. The first inequality gives $a \leq 1$ and the second one implies $a^4 - 2a^2 + a \leq 0 \iff a^3 - 2a + 1 \leq 0 \iff (a-1)(a^2+a-1) \leq 0$. Since $a \leq 1$, we obtain $a^2+a-1 \geq 0$.

The solution of this inequality is $a \in \left[\frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right]$. So we obtain $a \in \left[1, \frac{-1+\sqrt{5}}{2}\right]$.

**Problem 10.2.** Let $CH$ and $CM$ be an altitude and a median in a non-obtuse $\triangle ABC$. Let the bisector of angle $BAC$ meet $CH$ and $CM$ at points $P$ and $Q$, respectively. If $\angle ABP = \angle PBQ = \angle QBC$, prove that:

a) $\triangle ABC$ is a right triangle;

b) $BP = 2CH$.




**Solution:** a) Let $R = BC \cap AP$, $T = AC \cap BP$ and $S = AC \cap BQ$. Denote $AB = c$, $BC = a$, $CA = b$. It is easy to see that $P$ lies between $A$ and $Q$ (otherwise $\angle ABP > \angle PBQ$). It follows from Ceva's Theorem for point $P$ that:

$$(1) \qquad \frac{AH}{HB} \cdot \frac{BR}{RC} \cdot \frac{CT}{TA} = 1 \iff \frac{b\cos\alpha}{a\cos\beta} \cdot \frac{c}{b} \cdot \frac{S_{BTC}}{S_{ABT}} = 1$$

$$\iff \frac{b\cos\alpha}{a\cos\beta} \cdot \frac{c}{b} \cdot \frac{BT \cdot a \cdot \sin\frac{2\beta}{3}}{BT \cdot c \sin\frac{\beta}{3}} = 1 \iff \frac{\cos\alpha}{\cos\beta} \cdot \frac{\sin\frac{2\beta}{3}}{\sin\frac{\beta}{3}} = 1$$

$$\iff \frac{\cos\alpha}{\cos\beta} \cdot \frac{2\sin\frac{\beta}{3}\cos\frac{beta}{3}}{\sin\frac{\beta}{3}} = 1 \iff \frac{\cos\alpha}{\cos\beta} = \frac{1}{2\cos\frac{\beta}{3}}.$$

It follows from Ceva's Theorem for point $Q$ that:

(2)

$$\frac{AM}{MB}\cdot\frac{BR}{RC}\cdot\frac{CS}{SA}=1 \iff \frac{c}{b}\cdot\frac{S_{BSC}}{S_{ABS}}=1 \iff \frac{c}{b}\cdot\frac{BS\cdot a\cdot\sin\frac{\beta}{3}}{BS\cdot c\sin\frac{2\beta}{3}}=1$$

$$\iff \frac{c}{b}\cdot\frac{a\sin\frac{\beta}{3}}{c\sin\frac{2\beta}{3}}=1 \iff \frac{a\sin\frac{\beta}{3}}{2b\sin\frac{\beta}{3}\cos\frac{\beta}{3}}=1 \iff \frac{a}{b}=2\cos\frac{\beta}{3}.$$

Now (1) and (2) imply $\dfrac{\cos\alpha}{\cos\beta}=\dfrac{b}{a}$. From the Sine Law we obtain $\dfrac{\sin\beta}{\sin\alpha}=\dfrac{b}{a}$, so $\dfrac{\cos\alpha}{\cos\beta}=\dfrac{\sin\beta}{\sin\alpha} \iff \sin 2\alpha=\sin 2\beta$. If $\alpha=\beta$, then the triangle is isosceles and therefore $P\equiv Q$, implying that $\angle ABP=\angle PBQ=\angle QBC=0^\circ$, which is impossible. Thus $\alpha+\beta=90^\circ$ and therefore $\angle ACB=90^\circ$.

b) It follows from $\triangle BCS$ that $\cos\dfrac{\beta}{3}=\dfrac{a}{BS}$. Combining the above with (2) gives $2b=BS$. Note that $\triangle ABC\sim\triangle BHC$, which implies $\dfrac{BP}{CH}=\dfrac{BS}{AC}=2$. Therefore $BP=2CH$.

**Problem 10.3.** Let $A$ be a set of natural numbers with no zeroes in their decimal representation. It is known that if $a=\overline{a_1a_2\dots a_k}\in A$, then $b=\overline{b_1b_2\dots b_k}$, where $b_j, 1\le j\le k$ is the remainder of $3a_j$ modulo 10, belongs to $A$ and the sum of the digits of $b$ equals the sum of the digits of $a$.

a) Prove that the sum of the digits of a $k$-digit number in $A$ equals $5k$.

b) Find the smallest $k$-digit number which could be an element of $A$.

**Solution:** a) Let $a = \overline{a_1a_2\dots a_k}$ be a $k$-digit number from $A$, the sum of whose digits is $S$.

Consider the following numbers: $b = \overline{b_1b_2\dots b_k}, c = \overline{c_1c_2\dots c_k}$ and $d = \overline{d_1d_2\dots d_k}$, where $b_j, 1 \le j \le k$ is the remainder of $3a_j$ modulo 10, $c_j, 1 \le j \le k$ is the remainder of $3b_j$ modulo 10 and $d_j, 1 \le j \le k$ is the remainder of $3c_j$ modulo 10.

By the conditions of the problem all $b$, $c$ and $d$ belong to $A$. Further

$$(1) \qquad S = \sum_{j=1}^{k} a_j = \sum_{j=1}^{k} b_j = \sum_{j=1}^{k} c_j = \sum_{j=1}^{k} d_j.$$

Direct verification shows that for fixed $j$ the sum $a_j + b_j + c_j + d_j$ is equal to 20 (e. g., if $a_j = 3$, then $b_j = 9, c_j = 7, d_j = 1$ and therefore $a_j + b_j + c_j + d_j = 20$). It follows now from (1) that $4S = \sum_{j=1}^{k} a_j + \sum_{j=1}^{k} b_j + \sum_{j=1}^{k} c_j + \sum_{j=1}^{k} d_j = \sum_{j=1}^{k}(a_j + b_j + c_j + d_j) = 20k$. Therefore $S = 5k$, *Q. E. D.*

b) We shall prove that the required number is $a = \overline{a_1a_2\dots a_{2t}}$, where $a_1 = 1, a_2 = 1, \dots, a_t = 1, a_{t+1} = 9, a_{t+2} = 9, \dots, a_{2t} = 9$ if $k = 2t$ and $b = \overline{b_1b_2\dots b_{2t+1}}$, where $b_1 = 1, b_2 = 1, \dots, b_t = 1, b_{t+1} = 5, b_{t+2} = 9, \dots, b_{2t+1} = 9$ if $k = 2t + 1$. It is easy to see that $a$ and $b$ could be elements of a set having the required property.

Let $k = 2t$ and suppose there exists $c = \overline{c_1c_2\dots c_{2t}} \in A$ such that $c < a$. Since there are no zeroes among the digits of $c$, we obtain $c_1 = c_2 = \dots = c_t = 1$. But it follows from a) that the sum of the digits of $c$ is $5k = 10t$. The last is possible only if

$c_{t+1} = c_{t+2} = \ldots = c_{2t} = 9$. Hence $c = a$, a contradiction with the choice of $c$.

Similarly, suppose $k = 2t + 1$ and there exists $c = \overline{c_1 c_2 \ldots c_{2t+1}} \in A$ such that $c < b$. Since there are no zeroes among the digits of $c$ we obtain $c_1 = c_2 = \ldots = c_t = 1$. But it follows from a) that the sum of the digits of $c$ is $5k = 10t + 5$. The latter is possible only if $c_{t+1} \geq 5$ and since $c < b$, it follows that $c_{t+1} = 5$. It is easy to see now that $c_{t+1} = c_{t+2} = \ldots = c_{2t} = 9$. Hence $c = b$, a contradiction with the choice of $c$.

**Problem 11.1.** Given the sequence $a_n = n + a\sqrt{n^2+1}$, $n = 1, 2, \ldots$, where $a$ is a real number:

a) Find the values of $a$ such that the sequence $\{a_n\}_{n=1}^{\infty}$ is convergent.

b) Find the values of $a$ such that the sequence $\{a_n\}_{n=1}^{\infty}$ is monotone increasing.

**Solution:** a) If $a = -1$, the sequence $\{a_n\}_{n=1}^{\infty}$ is convergent because $a_n = n - \sqrt{n^2+1} = \dfrac{-1}{n + \sqrt{n^2+1}} = -\dfrac{1}{n}\dfrac{1}{1+\frac{1}{n^2}} \to 0$ when $n \to \infty$. Conversely, let the sequence $\{a_n\}_{n=1}^{\infty}$ be convergent. Since $a_n = n - \sqrt{n^2+1} + (a+1)\sqrt{n^2+1}$, we get that the sequence $(a+1)\sqrt{n^2+1}$ is also convergent. Since $\sqrt{n^2+1} \to \infty$ when $n \to \infty$, it follows that $a + 1 \neq 0$, so $a = -1$.

b) Let $\{a_n\}_{n=1}^{\infty}$ be a monotone increasing sequence, i. e., $a_{n+1} \geq$

$a_n$ for any $n$. This inequality is equivalent to

$$(\star) \qquad \frac{a(2n+1)}{\sqrt{(n+1)^2+1}+\sqrt{n^2+1}} \geq -1.$$

Since

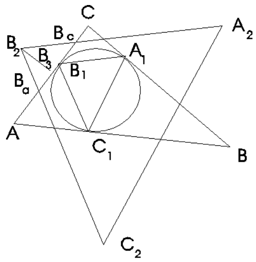
$$\lim_{n\to\infty} \frac{2n+1}{\sqrt{(n+1)^2+1}+\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{2+\frac{1}{n}}{\sqrt{(1+\frac{1}{n})^2+\frac{1}{n^2}}+\sqrt{1+\frac{1}{n^2}}} = 1$$

it follows from $(\star)$ that $a \geq -1$.

Conversely, let $a \geq -1$. It follows from $\frac{2n+1}{\sqrt{(n+1)^2+1}+\sqrt{n^2+1}} < \frac{2n+1}{n+1+n} = 1$ that $(\star)$ holds true so the sequence $\{a_n\}_{n=1}^{\infty}$ is increasing. The required values of $a$ are $a \in [-1, +\infty)$.

**Problem 11.2.** Given a $\triangle ABC$ with circumcentre $O$ and circumradius $R$. The incircle of $\triangle ABC$ is of radius $r$ and touches the sides $AB, BC$ and $CA$ in the points $C_1, A_1$ and $B_1$. Let the lines determined by the midpoints of the segments $AB_1$ and $AC_1$, $BA_1$ and $BC_1$, $CA_1$ and $CB_1$ meet at points $C_2, A_2$ and $B_2$. Prove that the circumcircle of $\triangle A_2B_2C_2$ is of centre $O$ and radius $R + \frac{r}{2}$.

**Solution:** We show first that the projection $B_3$ of $B_2$ on $AC$ is the midpoint of $AC$. Let $B_a$ and $B_c$ be the midpoints of $AB_1$ and $CB_1$. We shall use the standard notation for the elements of $\triangle ABC$.




We obtain $\dfrac{B_aB_3}{B_2B_3} = \operatorname{tg}\dfrac{\alpha}{2} = \dfrac{r}{p-a}$ and $\dfrac{B_cB_3}{B_2B_3} = \operatorname{tg}\dfrac{\gamma}{2} = \dfrac{r}{p-c}$, so $\dfrac{B_aB_3}{B_cB_3} = \dfrac{p-c}{p-a}$. Since $B_aB_c = \dfrac{b}{2}$, it follows that $B_aB_3 = \dfrac{p-c}{2} = \dfrac{CB_c}{2}$ and $B_cB_3 = \dfrac{p-a}{2} = \dfrac{AB_a}{2}$, which gives $AB_3 = CB_3$. Therefore $B_2O = B_2B_3 + B_3O = \dfrac{(p-c)(p-a)}{2r} + R\cos\beta$.

We shall show that the above expression equals $R+\dfrac{r}{2}$ and by analogy $A_2O = C_2O = R + \dfrac{r}{2}$, which will complete the proof.

We obtain that $\dfrac{(p-c)(p-a)}{2r} + R\cos\beta = \dfrac{r}{2} + R \iff \dfrac{S}{2(p-b)} - \dfrac{S}{2p} = R(1-\cos\beta) \iff \dfrac{Sb}{2p(p-b)} = 2R\sin^2\dfrac{\beta}{2} \iff \dfrac{r}{p-b} = \dfrac{4R}{b}\sin^2\dfrac{\beta}{2} \iff \operatorname{tg}\dfrac{\beta}{2} = \dfrac{2\sin^2\frac{\beta}{2}}{\sin\beta} \iff \sin\beta = 2\sin\dfrac{\beta}{2}\cos\dfrac{\beta}{2}$, which is a true equality.

**Problem 11.3.** Find the smallest natural number $n$ such that the sum of the squares of its divisors (including 1 and $n$) equals $(n+3)^2$.

**Solution:** It is clear that $n$ has at least three divisors and let $1 < d_1 < d_2 < \cdots < d_k < n$ be those different from 1 and $n$. The conditions of the problem imply

$$(\star) \qquad d_1^2 + d_2^2 + \cdots + d_k^2 = 6n + 8.$$

Let $n = p^\alpha$, where $p$ is a prime number. It follows now from $(\star)$ that $p^2 + p^4 + \cdots + p^{2\alpha-2} = 6p^\alpha + 8$, so $p \backslash 8$ and therefore $p = 2$. The above equality implies $1 + p^2 + p^4 + \cdots + p^{2\alpha-4} = 6p^{\alpha-2} + 2$, which is impossible.

Therefore $k \neq 1, 3, 5$, because otherwise the number of divisors of $n$ equals 3, 5, 7, i. e., $n = p^2$, $n = p^4$ or $n = p^6$, where $p$ is a prime number. Suppose that $k \geq 6$. Since $d_i d_{k-i} = n$, it follows from $(\star)$ that $(d_{k-1} - d_1)^2 + (d_{k-2} - d_2)^2 + (d_{k-3} - d_3)^2 \leq 8$. The last inequality is impossible, since the numbers $d_{k-1} - d_1, d_{k-2} - d_2$ and $d_{k-3} - d_3$ are distinct (if for example $d_{k-1} - d_1 = d_{k-2} - d_2 = A$, then $d_1(A + d_1) = d_2(A + d_2)$, so $d_1 = d_2$). We conclude now that $k = 2$ or $k = 4$.

Assume $k = 4$. Then $n$ has 6 divisors and thus $n$ is of the form $n = p \cdot q^2$, where $p$ and $q$ are distinct prime numbers. ($n$ is not of the form $n = p^5$). It follows from $(\star)$ that

$$(\star\star) \qquad p^2 + q^2 + q^4 + p^2q^2 = 6pq^2 + 8.$$

If $q \geq 5$, then $q^4 + p^2q^2 \geq 2pq^3 \geq 10pq^2 > 6pq + 8$ and therefore $q = 2$ or $q = 3$. Direct verification shows that inequality $(\star\star)$ is impossible. Thus $k = 2$ and hence $n = pq$, where $p$ and $q$ are distinct prime numbers such that

$$p^2 + q^2 = 6pq + 8.$$

Since $q \backslash p^2 - 8$, it is easy to see that if $p \leq 17$ then $p = 7$, $q = 41$ and $n = 287$. Since $17^2 = 289 > 287$, we conclude that the smallest $n$ with the required property is $n = 287$.